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Journal of Functional Analysis 214 (2004) 40–73
JOURNAL OF Functional Analysis
http://www.elsevier.com/locate/jfa

Compactness via symmetrization

Almut Burchard[a,*] and Yan Guo[b]

[a] *Department of Mathematics, University of Virginia, Charlottesville, VA 22902, USA*
[b] *Division of Applied Mathematics, Brown University, Providence, RI 02912, USA*

Received 17 June 2003; revised 9 January 2004; accepted 1 April 2004

Communicated by H. Brezis

Abstract

Consider two types of translation-invariant functionals $\mathscr{I}$ and $\mathscr{J}$ on $\mathbb{R}^m$, and a sequence of functions f_n whose corresponding symmetric rearrangements f_n^* are convergent. We show that f_n themselves converge up to translations if either $\lim_{n\to\infty}\mathscr{I}(f_n) = \lim_{n\to\infty}\mathscr{I}(f_n^*)$ or $\lim_{n\to\infty}\mathscr{J}(f_n) = \lim_{n\to\infty}\mathscr{J}(f_n^*)$. These compactness results lead to applications in variational problems and stability problems in stellar dynamics.

Keywords: Symmetric rearrangement; Concentration compactness; Dynamical stability; Hardy-Littlewood-Sobolev inequality; Sobolev inequality; Euler-Poisson system; Vlasov-Poisson system; Stellar dynamics

1. Introduction

1.1. Description of the problem

Lack of compactness is the main analytical difficulty in the study of functionals on unbounded domains. Ever since the Strauss radial lemma [34] (see also [35]), it has been well known that symmetry plays an important role in understanding the compactness in such problems. For many symmetric functionals, the existence of minimizers can be established by first restricting the problem to radially symmetric functions with the help of a rearrangement inequality, and then using the additional compactness of symmetric functions to find a convergent minimizing sequence.

*Corresponding author. Fax: +434-982-3084.
E-mail addresses: burchard@virginia.edu (A. Burchard), guoy@cfm.brown.edu (Y. Guo).

0022-1236/$ - see front matter
doi:10.1016/j.jfa.2004.04.005

Particular examples where this strategy has been used are the determination of the sharp constants in the Sobolev and Hardy–Littlewood–Sobolev inequalities [1,22,37], and the determination of ground states [21]. It is also known that certain dynamical stability problems can be reduced to the study of related variational problems [12]. Here, it is the compactness of arbitrary minimizing sequences, not just the existence of minimizers, that plays the key role. In a series of famous papers [25,26], Lions introduced a general abstract concentration compactness principle which has lead to many applications. It should be pointed out that in order to apply this principle to establish compactness for a specific problem, some additional analysis is usually needed.

In a series of recent investigations of stable galaxy configurations and gaseous stars [14–18,30,32], a splitting trick is combined with the crucial scaling property of the energy functional to establish compactness of all symmetric minimizing sequences. This allows to construct symmetric steady states, and to show that they are dynamically stable under symmetric perturbations, thus resolving a problem that had been open for a long time. In order to show stability among all possible perturbations, an argument in the spirit of the concentration compactness principle was employed to allow for possible translations.

The objective of this article is to closely examine the role of translations for minimizing sequences via elementary knowledge of their symmetrizations. We demonstrate that the difference between a minimizing sequence and the corresponding sequence of symmetrized functions is characterized by appropriate translations. In many cases, this implies that every minimizing sequence converges strongly modulo scalings and translations. Besides the interest of our results in classical analysis, this characterization also suggests a practical two-step procedure for proving compactness on an unbounded domain: *Step 1:* Show compactness of all symmetric minimizing sequences. This implies the existence of minimizers; it is also a necessary ingredient in the proof that these minimizers are dynamically stable under symmetric perturbations. *Step 2:* Show compactness up to translations for general minimizing sequences, assuming that their symmetrizations are compact. This implies dynamical stability under more general perturbations. The main part of this article is devoted to Step 2 for two classes of functionals that appear in many applications of the concentration compactness principle. We hope that our approach can give another perspective on concentration compactness for symmetric functionals.

1.2. Main results

The first class of functionals we consider is given by convolution integrals of the form

$$\mathcal{I}(f) = \int \int f(x)K(x-y)f(y)\,dx\,dy, \tag{1.1}$$

where $K \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$ is a nonnegative symmetrically decreasing function on $\mathbb{R}^m$.

Riesz' rearrangement inequality says that convolution integrals generally increase under symmetrically decreasing rearrangement [4,33], in particular

$$\mathscr{I}(f) \leqslant \mathscr{I}(f^*). \tag{1.2}$$

Here, f is a nonnegative measurable function that vanishes at infinity, and f^* is its symmetrically decreasing rearrangement. If either K or f^* is known to be strictly symmetrically decreasing, and $\mathscr{I}(f^*) < \infty$, then equality can occur only if f is a translate of f^* [8,21]. The second class consists of gradient integrals of the form

$$\mathscr{J}(f) = \int F(|\nabla f|)\, dx, \tag{1.3}$$

where F is an increasing convex function on $\mathbb{R}^+$ with $F(0) = 0$. It is well known that

$$\mathscr{J}(f) \geqslant \mathscr{J}(f^*) \tag{1.4}$$

for every nonnegative measurable function f on $\mathbb{R}^m$ that vanishes at infinity. If F is strictly convex, $\mathscr{J}(f^*) < \infty$, and the distribution function of f is absolutely continuous, then equality in Eq. (1.4) occurs only when f is a translate of f^* [7].

We are interested in applying these rearrangement inequalities to sequences of functions. Let f_n be a sequence of nonnegative functions on $\mathbb{R}^m$ that vanish at infinity, and let g be a symmetrically decreasing function. We make the additional assumptions that K is positive, strictly symmetrically decreasing, and defines a positive definite integral kernel on $\mathbb{R}^m$, and that F is strictly increasing convex with $F(0) = 0$. Then it is easy to see that both inequalities are preserved under taking limits: Using the continuity of $\mathscr{I}$ with respect to the norm defined by the positive definite quadratic form $\mathscr{I}$, we clearly have

$$\lim_{n\to\infty} \mathscr{I}(f_n^* - g) = 0 \quad \Rightarrow \quad \overline{\lim_{n\to\infty}}\, \mathscr{I}(f_n) \leqslant \lim_{n\to\infty} \mathscr{I}(f_n^*) = \mathscr{I}(g). \tag{1.5}$$

Likewise, combining Eq. (1.4) with Fatou's lemma shows that

$$\lim_{n\to\infty} \mathscr{J}(f_n^* - g) = 0 \quad \Rightarrow \quad \underline{\lim_{n\to\infty}}\, \mathscr{J}(f_n) \geqslant \underline{\lim_{n\to\infty}}\, \mathscr{J}(f_n^*) \geqslant \mathscr{J}(g). \tag{1.6}$$

Setting $f_n \equiv f$ and $g = f^*$, we recover the rearrangement inequalities in Eqs. (1.2) and (1.4). Our main result is that equality in either Eq. (1.5) or (1.6) implies, under suitable assumptions on K, F, and g, that the sequence f_n converges to g modulo translations.

Theorem 1. *Let $\mathscr{I}$ be a convolution functional as given in Eq.* (1.1), *where K is a strictly symmetrically decreasing function that defines a positive definite kernel on $\mathbb{R}^m$. Let g be a symmetrically decreasing function on $\mathbb{R}^m$ with $0 < \mathscr{I}(g) < \infty$, and let $\{f_n\}_{n\geqslant 1}$ be a sequence of nonnegative functions on $\mathbb{R}^m$ which vanish at infinity, with symmetrically decreasing rearrangements f_n^*. Assume that the sequence of* *rearrangements f_n^* approaches g in the sense that*

$$\lim_{n\to\infty} \mathscr{I}(f_n^* - g) = 0. \tag{1.7}$$

If the values of the functional converge to $\mathscr{I}(g)$ along the sequence,

$$\lim_{n\to\infty} \mathscr{I}(f_n) = \mathscr{I}(g), \tag{1.8}$$

then there exists a sequence of translations T_n on $\mathbb{R}^m$ such that

$$\lim_{n\to\infty} \mathscr{I}(T_n f_n - g) = 0.$$

The positive definiteness of K ensures that $\mathscr{I}(f-g)=0$ only for $f=g$. The assumption can be dropped if $\mathscr{I}(\cdot)$ is replaced by $\mathscr{I}(|\cdot|)$ in the assumptions and conclusions of the theorem. The classical equality statement for Eq. (1.2) is recovered by taking $f_n \equiv f$ and $g = f^*$.

Theorem 2. *Let $\mathscr{J}$ be a gradient functional of the form in Eq.* (1.3), *where F is a convex, strictly increasing function on $\mathbb{R}^+$ with $F(0)=0$. Let g be a symmetrically decreasing function on $\mathbb{R}^m$ that vanishes at infinity and satisfies $0<\mathscr{J}(g)<\infty$, and let $\{f_n\}_{n\geqslant 1}$ be a sequence of nonnegative measurable functions on $\mathbb{R}^m$ which vanish at infinity, with symmetrically decreasing rearrangements f_n^*. Assume that the symmetrically decreasing rearrangements f_n^* approach g in the sense that*

$$\lim_{n\to\infty} \mathscr{J}(f_n^* - g) = 0 \tag{1.9}$$

and that the values of the functional along the sequence converge to $\mathscr{J}(g)$,

$$\lim_{n\to\infty} \mathscr{J}(f_n) = \mathscr{J}(g). \tag{1.10}$$

Then the following statements hold:

1. *If F is strictly convex and the distribution function of g is absolutely continuous on the interval where it is finite and positive, then there exists a sequence of translations T_n on $\mathbb{R}^m$ such that*

$$\lim_{n\to\infty} \mathscr{J}\left(\frac{1}{2}(T_n f_n - g)\right) = 0.$$

2. *If $F(t)=t$, then there exists a sequence of translations T_n such that $T_n f_n$ is compact in $L^{\frac{m}{m-1}}(\mathbb{R}^m)$ and $\nabla(T_n f_n)$ is tight in $L^1(\mathbb{R}^m)$. If $f\in BV$ is a limit of a convergent subsequence, then $f^*=g$, and all level sets of f are balls.*

The purpose of the factor $1/2$ in the first conclusion of the theorem is to guarantee that the infimum is finite. The factor can be dropped under additional assumptions on F, in particular if $F(t) = t^p$ for some $p > 1$ or if F is linearly bounded. The equality statement for Eq. (1.4) due to Brothers and Ziemer [7] is again recovered by setting $f_n \equiv f$ and $g = f^*$.

In many applications to variational problems, assumptions (1.8) and (1.10) hold naturally for minimizing sequences, while assumptions (1.7) and (1.9) are related to compactness for symmetric minimizing sequences. Theorems 1 and 2 provide weak bounds on the asymmetry of a function in terms of the symmetrization deficit $\mathscr{I}(f^*) - \mathscr{I}(f)$ or $\mathscr{J}(f) - \mathscr{J}(f^*)$: Setting $f_n^* = g$ for all n, we see that the symmetrization deficit can be small only when f_n is close to a translate of g.

1.3. Description of the proofs

Mathematically, our results are inspired by so-called *asymmetry* inequalities, which estimate the difference between a function or a body and a symmetric one by a related geometric quantity. Classical examples are the Bonnesen-style isoperimetric inequalities, which give lower bounds on the excess perimeter of a planar set, as compared with the disc of the same area, in terms of geometric quantities such as the in-radius [3] (see [29]). The most powerful result in that direction is a quantitative isoperimetric inequality due to Hall [19], which bounds the symmetric difference between a measurable set and a (suitably translated) ball in terms of the isoperimetric deficit (a recent application of this result appears in [36]). Related statements have been proved for the logarithmic capacity in two dimensions and for the capacity of convex sets in higher dimensions [20]. We are not aware of estimates for the difference between the two sides of Riesz' rearrangement inequality in the literature, even though such estimates are readily obtained for the simpler two-point rearrangement [10,28]. We expect that asymmetry inequalities should hold for large classes of symmetric functionals, including the Coulomb electrostatic energy.

Our strategy for the proofs of Theorems 1 and 2 is as follows. We first write each function as the sum of a bounded function supported on a set of finite volume, and a function whose contribution to the functional is negligible (Section 2.2). To ensure that this decomposition commutes with translations and rearrangements, we use a technique closely related to the layer-cake principle [24, Theorem 3.9]. In the second step, we consider the symmetrization deficits $\mathscr{I}(f^*) - \mathscr{I}(f)$ and $\mathscr{J}(f) - \mathscr{J}(f^*)$ for a bounded function whose support has finite volume (Sections 3.1 and 4.1). We show that a function with a small symmetrization deficit must be almost supported on a suitably translated ball whose size we control (Lemmas 3.1 and 4.2). This is a key step that provides some basic compactness. It has the role that Lieb's compactness lemma [23] has played in many minimization problems (see, for example, [6,27]). In the third step (Sections 3.2 and 4.2), we pick subsequences that converge weakly up to translations, and identify their weak limits with the help of the classical equality statements for the rearrangement inequalities in Eqs. (1.2) and (1.4). This step is motivated by the characterization of the missing term in Fatou's lemma [5] (see [24, Theorem 1.9]). The proof is completed in Sections 3.3 and 4.3 by combining the three steps. In the final section, we discuss some recent and classical applications.

2. Preliminaries

2.1. Definitions and notation

Let f be a nonnegative measurable function on $\mathbb{R}^m$. We say that f *vanishes at infinity*, if for every $t>0$, the level set $\{x \in \mathbb{R}^m \mid f(x) > t\}$ has finite measure. The *distribution function* of f is given by

$$\mu(t) = \int \mathbf{1}_{f(x)>t}\, dx.$$

The *symmetric decreasing rearrangement*, $f^{*\prime}$ of f is the symmetrically decreasing, lower semicontinuous function equimeasurable to f,

$$f^*(x) = \sup\{t>0 \mid \mu(t) > \omega_m |x|^m\},$$

where ω_m is the volume of the unit ball in $\mathbb{R}^m$.

2.2. Decomposition into layers

In the proofs of Theorems 1 and 2, we find it useful to write a given function f as a sum of *layers*, $f = f^b + f^u$, where the middle layer

$$f^b = [\min\{f, f^*(R^{-1})\} - f^*(R)]_+ \tag{2.1}$$

is bounded and has level sets of bounded measure, and the sum of the top and bottom layers

$$f^u = f - f^b = \min\{f, f^*(R)\} + [f - f^*(R^{-1})]_+ \tag{2.2}$$

will be negligible for R sufficiently large (see Fig. 1). If f is equimeasurable to g, then f^b and f^u are equimeasurable to g^b and g^u, respectively. In particular, this

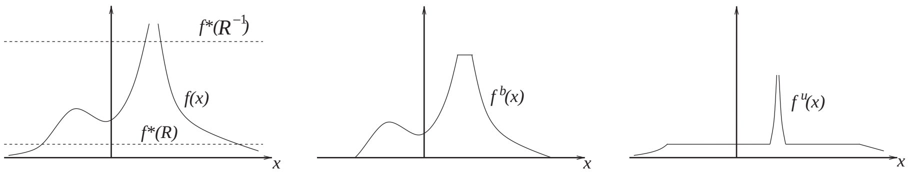

Fig. 1. Construction of the layers f^b and f^u.

decomposition commutes with rearrangements and translations. The following lemma will be used to obtain uniform bounds on the sequence f_n^b.

Lemma 2.1. *Let $\mathcal{I}$ be a convolution functional of the form given in Eq.* (1.1) *with K symmetrically decreasing and not identically zero, and let $\mathcal{J}$ be a gradient functional as defined in Eq.* (1.3) *with F convex, strictly increasing, and $F(0)=0$. Fix $R>1$ and $I_0, J_0>0$. For a nonnegative measurable function f that vanishes at infinity, define the middle layer f^b by Eq.* (2.1). *There exist constants $C_1(R, I_0)$ and $C_2(R, J_0)$ such that*

$$||f^b||_\infty \leqslant C_1(R, I_0)$$

for all functions f with $\mathcal{I}(f^*) \leqslant I_0$, and

$$||f^b||_\infty \leqslant C_2(R, J_0)$$

for all f with $\mathcal{J}(f^*) \leqslant J_0$.

Proof. Since $||f^b||_\infty$ increases with R, it suffices to prove the claim for large values of R. For the first claim, we use the fact that K and f^* are symmetrically decreasing to estimate

$$\begin{aligned} \mathcal{I}(f^*) &\geqslant \int \int_{|x|,|y|<R^{-1}} f^*(x) K(x-y) f^*(y)\, dx\, dy \\ &\geqslant K(2R^{-1})(m\omega_m R^{-m} f^*(R^{-1}))^2 \\ &\geqslant K(2R^{-1})(m\omega_m R^{-m} ||f^b||_\infty)^2, \end{aligned}$$

where ω_m is the volume of the unit ball in $\mathbb{R}^m$.
In the last line, we have used that $||f^b||_\infty \leqslant f(R^{-1})$ by construction. The first claim follows since $K(2R^{-1})>0$ for R sufficiently large by assumption. To see the second claim, let ϕ be the function on $\mathbb{R}^+$ determined by $|\nabla f^*(x)| = \phi(|x|)$, and compute in polar coordinates

$$\begin{aligned} \mathcal{J}(f^*) &\geqslant \int_{R^{-1}}^{R} F(\phi(r))\, m\omega_m r^{m-1}\, dr \\ &\geqslant m\omega_m R^{1-m}(R-R^{-1})\, F\left(\int_{R^{-1}}^{R} \phi(r) \frac{dr}{R-R^{-1}}\right) \\ &\geqslant m\omega_m R^{2-m}\, F\left(\frac{||f^b||_\infty}{R}\right). \end{aligned}$$

In the second step, we have estimated the factor r^{m-1} from below by R^{1-m}, then applied Jensen's inequality. Since $tF(x/t)$ is nonincreasing in t, we can replace $R - R^{-1}$ by R in the third step. The claimed bound on $||f^b||_\infty$ follows since F is strictly increasing. $\square$

It is easy to see that the assumptions in Eqs. (1.9) and (1.10) of Theorem 2 hold also for the middle layers f_n^b and g^b of the functions f_n and g:

Lemma 2.2. *Let* $\mathscr{J}$ *be a gradient functional of the form in Eq.* (1.3) *with* F *convex, strictly increasing, and* $F(0) = 0$, *and let* g *be a symmetrically decreasing function with* $\mathscr{J}(g) < \infty$. *Fix* $R > 1$, *and decompose* f_n, f_n^*, *and* g *into layers as in Eqs.* (2.1)–(2.2). If

$$\lim_{n\to\infty} \mathscr{J}(f_n^* - g) = 0,$$

then

$$\lim_{n\to\infty} \mathscr{J}(f_n^{*b} - g^b) = 0, \quad \lim_{n\to\infty} \mathscr{J}(f_n^{*u} - g^u) = 0.$$

If, additionally,

$$\lim_{n\to\infty} \mathscr{J}(f_n) = \mathscr{J}(g),$$

then

$$\lim_{n\to\infty} \mathscr{J}(f_n^b) = \mathscr{J}(g^b), \quad \lim_{n\to\infty} \mathscr{J}(f_n^u) = \mathscr{J}(g^u).5$$

Proof. Since

$$\nabla f^{*b}(x) = \nabla f^*(x)\, \mathbf{1}_{R^{-1} \leqslant |x| \leqslant R},$$

we can rewrite the first assumption as

$$\lim_{n\to\infty} \{\mathscr{J}(f_n^{*b} - g^b) + \mathscr{J}(f_n^{*u} - g^u)\} = 0,$$

which clearly implies that both summands converge to zero, as claimed. To see the second claim, we note that

$$\nabla f^b(x) = \nabla f(x)\, \mathbf{1}_{f^*(R) \leqslant f(x) \leqslant f^*(R^{-1})}$$

and rewrite the additional assumption as

$$\lim_{n\to\infty} \{(\mathscr{J}(f_n^b) - \mathscr{J}(g^b)) + (\mathscr{J}(f_n^u) - \mathscr{J}(g^u))\} = 0.$$

The claim follows since the limit of each summand is nonnegative by Eq. (1.6). $\square$

The corresponding statement holds for the functional $\mathscr{I}$ appearing in Theorem 1.

Lemma 2.3. *Let* $\mathscr{I}$ *be a convolution functional of the form in Eq.* (1.1) *with* K *positive definite and strictly symmetrically decreasing, and let* g *be a symmetrically* *decreasing function with* $\mathcal{I}(g) < \infty$. *Fix* $R > 1$, *and decompose* f_n, f_n^* *and* g *into layers as in Eqs.* (2.1) and (2.2). *If*

$$\lim_{n\to\infty} \mathcal{I}(f_n^* - g) = 0,$$

then

$$\lim_{n\to\infty} \mathcal{I}(f_n^{*b} - g^b) = 0, \quad \lim_{n\to\infty} \mathcal{I}(f_n^{*u} - g^u) = 0.$$

If, *additionally*,

$$\lim_{n\to\infty} \mathcal{I}(f_n) = \mathcal{I}(g),$$

then

$$\lim_{n\to\infty} \mathcal{I}(f_n^b) = \mathcal{I}(g^b), \quad \lim_{n\to\infty} \mathcal{I}(f_n^u) = \mathcal{I}(g^u).$$

The proof requires some auxiliary estimates. The first lemma provides three tail estimates for symmetrically decreasing functions g in terms of $\mathcal{I}(g)$.

Lemma 2.4. *If* K *and* g *are nonnegative and symmetrically decreasing*, *then*, *for any* $R > 0$,

$$\mathcal{I}(g) \geqslant K(2R)\left(\int_{|x|\leqslant R} g(x)\,dx\right)^2, \tag{2.3}$$

$$\mathcal{I}(g) \geqslant \left(\int_{|x|\geqslant 2R} g(x)K(|x|+R)\,dx\right)\left(\int_{|x|<R} g(x)\,dx\right). \tag{2.4}$$

Furthermore, *for every* $h \in L^1(\mathbb{R}^m)$ *supported in the ball* $|x| \leqslant R_0$, *and every* $\varepsilon > 0$ *there exists a number* $R > 0$ *which depends only on* K, R_0, *and* ε *such that*

$$\int_{|x|\geqslant R} g(x)K * h(x)\,dx \leqslant \varepsilon ||h||_1 \mathcal{I}(g)^{1/2}. \tag{2.5}$$

Proof. Eqs. (2.3) and (2.4) follow immediately from the fact that both K and g are nonnegative and symmetrically decreasing. To see the weak tail estimate in Eq. (2.5), we separate two cases. If

$$||g||_1 \leqslant \frac{(3/2)^{m-1}}{\varepsilon} \mathcal{I}(g)^{1/2},$$

then we have for $R > R_0$:

$$\int_{|x|\geqslant R} g(x)|K * h(x)|\,dx \leqslant ||h||_1||g||_1 K(R-R_0) \leqslant ||h||_1 \frac{(3/2)^{m-1}K(R-R_0)}{\varepsilon}\mathscr{I}(g)^{1/2}$$

and Eq. (2.5) follows by choosing R large enough such that $K(R-R_0)(3/2)^{m-1} \leqslant \varepsilon^2$. If, on the other hand,

$$\int_{|x|<R_1} g(x)\,dx > \frac{(3/2)^{m-1}}{\varepsilon}\mathscr{I}(g)^{1/2}$$

for some $R_1 \geqslant R_0$, then we estimate for $R \geqslant 4R_1$,

$$\int_{|x|\geqslant R} g(x)|K * h(x)|\,dx \leqslant ||h||_1 \int_{|x|\geqslant 4R_1} g(x)K(|x|-R_1)\,dx.$$

The integral on the right-hand side is bounded by

$$\begin{aligned}\int_{|x|\geqslant 4R_1} g(x)K(|x|-R_1)\,dx &\leqslant \int_{|x|\geqslant 2R_1} g(x)K(|x|+R_1)\left(\frac{|x|+2R_1}{|x|}\right)^{m-1} dx\\ &\leqslant (3/2)^{m-1}\frac{\mathscr{I}(g)}{\int_{|x|\leqslant R_1} g(x)\,dx}\\ &\leqslant \varepsilon\mathscr{I}(g)^{1/2}.\end{aligned}$$

In the first step, we have estimated $g(x) \leqslant g(|x|-2R_1)$ and changed variables in polar coordinates. Next, we have used that $|x|+2R_1 \leqslant (3/2)|x|$ and applied Eq. (2.4). Inserting the last inequality into the preceding equation again yields Eq. (2.5). $\square$

Lemma 2.5. *Let g_n be a sequence of be nonnegative, symmetrically decreasing functions on $\mathbb{R}^m$ which vanish at infinity, and decompose them into layers by Eqs.* (2.1) *and* (2.2) *for some $R>1$. Let g be a nonnegative, symmetrically decreasing function on* $\mathbb{R}^m$ such that $\mathscr{I}(g) < \infty$, *where $\mathscr{I}$ is defined by Eq.* (1.1) *with a strictly symmetrically decreasing, positive definite kernel K. If*

$$\lim_{n\to\infty} \mathscr{I}(g_n - g) = 0,$$

then

$$\lim_{n\to\infty} \mathscr{I}(g_n^b - g^b) = 0, \quad \lim_{n\to\infty} \mathscr{I}(g_n^u - g^u) = 0.$$

Proof. It suffices to establish that a subsequence of g_n converges to g pointwise almost everywhere; the claim then follows by applying Fatou's lemma to

$$\int\int \{[g_n(x)+g(x)][g_n(y)+g(y)] - [g_n^{\#}(x)-g^{\#}(x)][g_n^{\#}(y)-g^{\#}(y)]\}K(x-y)\,dx\,dy$$

for $\# = b, u$.

In order to prove pointwise convergence, we first notice that

$$\lim_{n\to\infty} \mathscr{I}(g_n) = \mathscr{I}(g)$$

by Eq. (1.5). By Cauchy–Schwarz, the assumption implies

$$\lim_{n\to\infty}\int\int g_n(x)K(x-y)h(y)\,dx\,dy = \int\int g(x)K(x-y)h(y)\,dx\,dy$$

for any function h with $\mathscr{I}(h)<\infty$. This means that $K * g_n$ converges to $K * g$ in the sense of distributions. The sequence g_n is uniformly bounded in L^1_{loc} by Eq. (2.3). Since the functions g_n are symmetrically decreasing, we can choose a subsequence (still denoted by g_n) such that

$$g_n \rightharpoonup a\delta_0 + g_0 \quad \text{in the sense of distributions, and}$$

$$g_n \to g_0 \quad \text{pointwise a.e.}$$

Here $a\geqslant 0$, δ_0 is the Dirac mass at the origin, and $g_0\geqslant 0$ is a symmetrically decreasing function with $\mathscr{I}(g_0)<\infty$. We need to show that $a=0$. To this end, fix any $h\in C_0^\infty$. Since $\sup_{n\geqslant 1}\mathscr{I}(g_n)<\infty$, Eq. (2.5) of Lemma 2.4 implies that there exists for each $\varepsilon>0$ a number $R>0$ such that

$$\sup_{n\geqslant 1}\int_{|x|\geqslant R} g_n(x)|K * h(x)|\,dx \leqslant \varepsilon.$$

It follows that

$$\begin{aligned}\int (K*g)h &= \lim_{n\to\infty}\int g_n(x)K*h(x)\,dx\\ &= \lim_{R\to\infty}\lim_{n\to\infty}\int_{|x|\leqslant R} g_n(x)K*h(x)\,dx\\ &= \int K*\{a\delta_0+g_0\}(x)h(x)\,dx,\end{aligned}$$

where we have used that $K * g_n$ and g_n converge in the sense of distributions. Since h is arbitrary, we conclude that

$$K*\{a\delta_0+g_0\} = K*g,$$

which implies that $a\delta_0 + g_0 = g$ by the positive definiteness of K, and the desired pointwise convergence follows. This completes the proof of the lemma. $\square$

Proof of Lemma 2.3. Applying Lemma 2.5 to the sequence f_n^* of symmetric decreasing rearrangements, we see that the first assumption implies the first claim. To see that the additional assumption implies the second claim, we note that

$$\lim_{n\to\infty} \mathscr{I}(f_n^b) \leqslant \mathscr{I}(g^b), \quad \lim_{n\to\infty} \mathscr{I}(f_n^u) \leqslant \mathscr{I}(g^u)$$

by Eq. (1.5). Similarly, using first Riesz' rearrangement inequality and then the continuity with respect to the norm defined by the positive definite quadratic form $\mathscr{I}$, we have

$$\overline{\lim_{n\to\infty}} \int\int f_n^b(x)K(x-y)f_n^u(y)\,dx\,dy \leqslant \int\int g^b(x)K(x-y)g^u(y)\,dx\,dy.$$

Adding these inequalities proves the second claim of the lemma. $\square$

3. Convolution integrals

3.1. Confinement to a ball

Lemma 3.1. *Consider the convolution functional $\mathscr{I}$ defined in Eq.* (1.1) *with some symmetrically decreasing, nonnegative integral kernel K. Let f be a nonnegative measurable function that vanishes at infinity, and assume that its symmetrically decreasing rearrangement f^* is supported on a ball of radius R_0 and satisfies $\mathscr{I}(f^*) < \infty$. Then there exists for any choice of $R_1 > 2R_0$ a translation T such that*

$$\mathscr{I}(f^*) - \mathscr{I}(f) \geqslant (K(2R_0) - K(R_1))\left(\int_{|x|>R_1} Tf(x)\,dx\right)^2.$$

Proof. We decompose the kernel as

$$K = [K - K(2R_0)]_+ + \min[K, K(2R_0)].$$

Since both summands are nonnegative and symmetrically decreasing, Riesz' rearrangement inequality implies

$$\mathscr{I}(f^*) - \mathscr{I}(f) \geqslant \int\int f^*(x)f^*(y)\min[K(x-y), K(2R_0)]\,dx\,dy$$

$$
\begin{aligned}
&- \int \int f(x) f(y) \min[K(x-y), K(2R_0)] \, dx \, dy \\
&\geqslant 0.
\end{aligned}
$$

The first integral on the right-hand side can be rewritten as

$$
\begin{aligned}
\int \int f^*(x) f^*(y) \min[K(x-y), K(2R_0)] &= \int f^*(x) f^*(y) K(2R_0) \, dx \, dy \\
&= \int f(x) f(y) K(2R_0) \, dx \, dy,
\end{aligned}
$$

where we have used that f is supported on the ball of radius R_0 in the first step, and the equimeasurability of f with f^* in the second. We obtain

$$
\begin{aligned}
\mathcal{I}(f^*) - \mathcal{I}(f) &\geqslant \int \int f(x) f(y) \{K(2R_0) - \min[K(2R_0), K(x-y)]\} \, dx \, dy \\
&\geqslant \{K(2R_0) - K(R_1)\} \int \int f(x) f(y) \mathbf{1}_{|x-y| \geqslant R_1} \, dx \, dy.
\end{aligned}
$$

Letting $h(y) = \int f(x) \mathbf{1}_{|x-y| \geqslant R_1} \, dx$, we deduce by the mean value theorem that there exists a point x_0 so that

$$
\int f(y) h(y) \, dy \geqslant h(x_0) \int f(y) \, dy.
$$

We have shown that

$$
\begin{aligned}
\mathcal{I}(f^*) - \mathcal{I}(f) &\geqslant \{K(2R_0) - K(R_1)\} \int f(y) \, dy \times \int f(x) \mathbf{1}_{|x-x_0| \geqslant R_1} \, dx \\
&\geqslant \{K(2R_0) - K(R_1)\} \left(\int f(x) \mathbf{1}_{|x-x_0| \geqslant R_1} \, dx \right)^2.
\end{aligned}
$$

Setting $Tf(x) = f(x + x_0)$ completes the proof. $\square$

3.2. *Identification of the limit*

Lemma 3.2. *Let f_n be a sequence of nonnegative functions in L^2, and let $\mathcal{I}$ be as in Eq.* (1.1), *with a nonnegative symmetrically decreasing kernel K. If $f_n \rightharpoonup f$* and *$f_n^* \rightharpoonup g$ weakly in L^2 for some functions f and g, then*

$$
\mathcal{I}(f) \leqslant \mathcal{I}(g).
$$

If K is strictly symmetrically decreasing and $\mathcal{I}(g) < \infty$, then equality implies that there exists a translation T such that $Tf = g$.

Proof. For any nonnegative function $h \in L^2$, we have

$$\int f(x)h(x)\,dx = \lim_{n\to\infty} \int f_n(x)h(x)\,dx \leqslant \lim_{n\to\infty} \int f_n^*(x)h^*(x)\,dx = \int g(x)h^*(x)\,dx.$$

Since f and f^* are equimeasurable, it follows from the bathtub principle that

$$\begin{aligned}\int_{|x|<R} f^*\,dx &= \sup_{A:\ Vol(A)=\omega_m R^m} \int_A f^*(x)\,dx \\ &= \sup_{A:\ Vol(A)=\omega_m R^m} \int_A f(x)\,dx \leqslant \int_{|x|<R} g(x)\,dx\end{aligned}$$

for any $R>0$. Applying the layer-cake principle we conclude that

$$\int f^*(x)h(x)\,dx \leqslant \int g(x)h(x)\,dx \tag{3.1}$$

for every symmetrically decreasing function h. If h is strictly symmetrically decreasing and the integrals are finite, then equality in Eq. (3.1) can occur only for $f^* = g$.

It follows with Riesz' rearrangement inequality that

$$\mathcal{I}(f) \leqslant \mathcal{I}(f^*) \leqslant \int f^*(x) K * g(x)\,dx \leqslant \mathcal{I}(g),$$

where we have applied Eq. (3.1) twice, first with $h = K * f^*$ and then with $h = K * g$. If K is strictly symmetrically decreasing, then equality in the Riesz rearrangement inequality implies that there exists a translation T such that $Tf = f^*$. Furthermore, since $K * f^*$ and $K * g$ are again strictly symmetrically decreasing, equality in the last step implies that $f^* = g$. $\square$

3.3. Proof of Theorem 1

Let f_n, g, and K be as in the statement of the theorem, and assume for the moment that the functions f_n are uniformly bounded, and that their symmetrically decreasing rearrangements f_n^* are supported in a ball of radius R. By Lemma 3.1, there exists a sequence of translations T_n such that

$$\int_{|x|\geqslant 3R} T_n f_n(x)\,dx \leqslant \left(\frac{\mathcal{I}(f_n^*) - \mathcal{I}(f_n)}{K(2R) - K(3R)}\right)^{1/2} \to 0 \quad (n\to\infty). \tag{3.2}$$

Since $||T_n f_n||_2^2 = ||f_n^*||_2^2$ is uniformly bounded, the sequence $T_n f_n$ is weakly compact in L^2, that is, there exists a subsequence, again denoted by f_n and a function f with

$$T_n f_n \rightharpoonup f \quad (n\to\infty) \tag{3.3}$$

weakly in L^2. In light of Lemma 3.2, the value $\mathcal{I}(f)$ is finite. Our goal is to show that $\mathcal{I}(T_n f_n - f) \to 0$ as $n \to \infty$. To this end, fix $\varepsilon > 0$, and split

$$K = K\mathbf{1}_{|x|<\varepsilon} + K\mathbf{1}_{|x|\geqslant\varepsilon} = K^s + K^c,$$

so that

$$\begin{aligned}\mathcal{I}(T_n f_n - f) = &\int_{|x|<3R} (T_n f_n - f)K^c * (T_n f_n - f)\,dx \\ &+ \int_{|x|\geqslant 3R} (T_n f_n - f)K^c * (T_n f_n - f)\,dx \\ &+ \int (T_n f_n - f)K^s * (T_n f_n - f)\,dx.\end{aligned}$$

The first integral on the right-hand side goes to zero, because the sequence $\{(K^c * T_n f_n)\mathbf{1}_{|x|<3R}\}_{n\geqslant 1}$ is compact in L^2 by the Hilbert–Schmidt theorem, and

$$(K^c * T_n f_n)\mathbf{1}_{|x|<3R} \to (K^c * f)\mathbf{1}_{|x|<3R} \quad (n \to \infty).$$

For the second integral, notice that

$$\int T_n f_n (K^c * T_n f_n)\mathbf{1}_{|x|\geqslant 3R} \leqslant ||K^c||_\infty ||f||_1 \int_{|x|\geqslant 3R} f_n(x)\,dx \to 0 \quad (n \to \infty)$$

by Eq. (3.2). The third integral is estimated by

$$\int T_n f_n(x) K^s * f_n(x)\,dx \leqslant ||f_n||_\infty ||f_n||_1 \int_{|x|\leqslant\varepsilon} K(x)\,dx$$

which can be made small by choosing ε small. We conclude that $\mathcal{I}(T_n f_n - f) \to 0$. Since $\mathcal{I}(f) = \mathcal{I}(g^*)$ by assumption, Lemma 3.2 implies that $T_0 f = g$ for some translation T_0. Thus we have shown that

$$\inf_T \mathcal{I}(Tf_n - g) \leqslant \mathcal{I}(T_0 T_n f_n - g) \to 0 \quad (n \to \infty)$$

at least along a suitable subsequence. Since the limit does not depend on the subsequence, this proves the claim in the special case that the rearrangements f_n^* are uniformly bounded and supported on a common ball.

Given a sequence of functions f_n, which satisfy the convergence assumptions of the theorem. If the functions f_n and g are not uniformly bounded or have level sets of large measure, we write them as a sum of layers, $f_n = f_n^b + f_n^u$ and $g = g^b + g^u$, according to Eqs. (2.1) and (2.2), where $R > 1$ is a large number that will be chosen below. By Cauchy–Schwarz, and using that Tf_n is equimeasurable with f_n, we can estimate

$$\inf_T \mathscr{I}(Tf_n - g) \leqslant 3\left\{\inf_T \mathscr{I}(Tf_n^b - g^b) + \mathscr{I}(f_n^u) + \mathscr{I}(g^u)\right\}. \tag{3.4}$$

By Lemma 2.1, the functions f_n^b are uniformly bounded, and by construction, their symmetric decreasing rearrangements are supported on the ball of radius R. By Lemma 2.3, the functions f_n^b satisfy the assumptions of the theorem as well, with g replaced by g^b. We have shown in the first part of the proof that

$$\lim_{n\to\infty} \inf_T \mathscr{I}(Tf_n^b - g^b) = 0.$$

Furthermore, by Lemma 2.3, we have

$$\lim_{n\to\infty} \mathscr{I}(f_n^u) = \mathscr{I}(g^u).$$

Taking limits in Eq. (3.4), we obtain

$$\overline{\lim_{n\to\infty}} \inf_T \mathscr{I}(Tf_n - g) \leqslant 6\mathscr{I}(g^u).$$

Since the right-hand side can be made arbitrarily small by choosing R large enough, this completes the proof.

4. Convex gradient functionals

4.1. *Confinement to a ball*

We begin with a lower bound for the isoperimetric deficit in terms of a volume integral. The following lemma can be obtained as a corollary of [19]; for the convenience of the reader, we give here a direct proof. Denote by $Vol(A)$ the m-dimensional Lebesgue measure of a set $A \subset \mathbb{R}^m$, and by $Per(A)$ its perimeter.

Lemma 4.1. *If $A \subset \mathbb{R}^m$ has finite perimeter, then*

$$\frac{Per(A) - Per(A^*)}{Per(A^*)} \geqslant \frac{\alpha_m}{R^{2m}} \int_A \int_A \mathbf{1}_{|x-y| \geqslant \beta_m R}\, dx\, dy, \tag{4.1}$$

where R is the radius of A^*, $\alpha_m = (2^{1/m} - 1)/(4m\omega_m^2)$, and $\beta_m = -4\sqrt{m}\log(1 - 2^{-1/m})$.

Proof. We will use a simplified version of Hall's argument to show that all but a fraction of the volume of A can be enclosed in a large box in $\mathbb{R}^m$, and use that to bound the integral in Eq. (4.1). Since the integral is bounded above by $(2^{1/m} - 1)/(4m) < 1/2$, we may assume without loss of generality that $(Per(A) -$

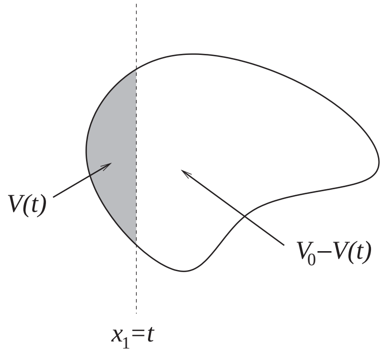

Fig. 2. Proof of Lemma 4.1. The perimeter of the entire set is at least as large as sum of the perimeters of two balls with volumes $V(t)$ and $V_0 - V(t)$, minus twice the area of the interface.

$Per(A^*))/Per(A^*) < 1/2$. Let $V(t)$ be the volume of A to the left of the hyperplane $x_1 = t$ (see Fig. 2). We assume that $V(0) = Vol(A)/2$, that is, half of the volume of A lies in the negative half-space. Applying the isoperimetric inequality to the parts of A on either side of the hyperplane $x_1 = t$, and subtracting twice the area of the interface, we obtain for the perimeter of A

$$Per(A) \geqslant m\omega_m \left\{ \left(\frac{V(t)}{\omega_m}\right)^{1-1/m} + \left(\frac{Vol(A) - V(t)}{\omega_m}\right)^{1-1/m} \right\} - 2V'(t).$$

Let $y(t) = V(t)/Vol(A)$ be the volume fraction of A to the left of the hyperplane $x_1 = t$. Using that $Vol(A) = \omega_m R^m$, $Per(A^*) = m\omega_m R^{m-1}$, and solving for y', we obtain

$$\frac{2R}{m} y'(t) \geqslant y^{1-1/m} + (1-y)^{1-1/m} - 1 - \frac{Per(A) - Per(A^*)}{Per(A^*)}.$$

Note that $y(0) = 1/2$ by our choice of coordinates. We next use the concavity of the function $u \to u^{1-1/m}$ to see that for $y \leqslant 1/2$

$$\frac{1 - (1-y)^{1-1/m}}{y} \leqslant \frac{1 - (1/2)^{1-1/m}}{1/2} = 2 - 2^{1/m} < 1.$$

Inserting the last equation into the previous one shows that

$$\frac{2R}{m} y'(t) \geqslant y^{1-1/m} - y \tag{4.2}$$

so long as

$$\frac{Per(A) - Per(A^*)}{(2^{1/m} - 1)\, Per(A^*)} \leqslant y \leqslant 1/2. \tag{4.3}$$

Since the right-hand side of Eq. (4.2) is strictly positive, we can separate variables and obtain by direct integration

$$t_2 - t_1 \leqslant 2R \int_{y_1}^{y_2} \frac{1}{y^{1-1/m} - y}\, dy = 2R\{-\log(1 - y_2^{1/m}) + \log(1 - y_1^{1/m})\},$$

provided Eq. (4.3) holds for $t \in (t_1, t_2)$. Plugging in $y_1 = \frac{Per(A) - Per(A^*)}{\{2^{1/m} - 1\} Per(A^*)} > 0$, $y_2 = 1/2$ and $t_2 = 0$, we conclude that all but a fraction y_1 of the volume of A lies to the right of the hyperplane $t = 2R \log(1 - 2^{-1/m})$. Repeating the argument for the right half of A and for the other $m - 1$ coordinate directions, we see that all but a fraction $2my_1$ of the volume of A is contained in a box of side length $-4R \log(1 - 2^{-1/m})$. Since the diameter of this box is $\beta_m R$, it follows that

$$\int_A \int_A \mathbf{1}_{|x-y| \geqslant \beta_m R}\, dx\, dy \leqslant 2(2my_1)\, Vol(A)^2 \leqslant \frac{R^{2m}}{\alpha_m} \frac{Per(A) - Per(A^*)}{Per(A^*)}$$

as claimed. □

Lemma 4.2. *Let F be a nondecreasing convex function on* $\mathbb{R}^+$ *with* $F(0) = 0$, *and define* $\mathcal{J}$ *by Eq.* (1.3). *Assume that f is a nonnegative function on* $\mathbb{R}^m$ *with* $\mathcal{J}(f) < \infty$, *whose symmetrically decreasing rearrangement* f^* *is supported in the ball of radius R. Then there exists a translation T such that, for any* $\varepsilon > 0$,

$$\mathcal{J}(f) - \mathcal{J}(f^*) \geqslant \frac{\alpha_m}{R^{2m} \mathcal{J}(f^*)} \left(\mathcal{J}(\min(f^*, \varepsilon)) \int_{|x| \geqslant \beta_m R} \mathbf{1}_{Tf(x) \geqslant \varepsilon}\, dx \right)^2,$$

where α_m *and* β_m *the constants from Lemma* 4.1.

Proof. The convexity of F implies, via the co-area formula and Jensen's inequality, that

$$\mathcal{J}(f) \geqslant \int_0^\infty Per(\{f > h\}) G(|r'(h)|)\, dh, \tag{4.4}$$

where $G(z) = zF(z^{-1})$ is a nonnegative, nonincreasing and convex function on $\mathbb{R}^+$, $r(h)$ is the radius of the ball $\{x \in \mathbb{R}^m | f^* > h\}$, and $r'(h)$ is its derivative from the left [7, Eqs. (33)–(35)]. (Note that the convexity of $F^{1/p}$ assumed there is obsolete, see [9, Proposition 4.1]). We set $G(|r'(h)|) = 0$ if h is a singular value of f^*. Since Eq. (4.4) is an identity when $f = f^*$, we have

$$\mathcal{J}(f) - \mathcal{J}(f^*) \geqslant \int_0^\infty [Per(\{f > h\}) - Per(\{f^* > h\})] G(|r'(h)|)\, dh.$$

Applying Lemma 4.1 to the integrand results in

$$\begin{aligned}\mathcal{J}(f) - \mathcal{J}(f^*) &\geqslant \alpha_m \int_0^\infty \frac{Per(\{f^* > h\})}{r(h)^{2m}} G(|r'(h)|) \int_{f(y)>h} \int_{f(x)>h} \mathbf{1}_{|x-y|\geqslant \beta_m R}\, dx\, dy\, dh \\ &\geqslant \frac{\alpha_m}{R^{2m}} \int \int \mathbf{1}_{|x-y|\geqslant \beta_m R} \int_0^{\min(f(x),f(y))} Per(\{f^* > h\}) G(|r'(h)|)\, dh\, dx\, dy.\end{aligned}$$

In the second step, we have exchanged the order of integration and used that $r(h) \leqslant R$ by our assumption on the support of f^*. To simplify notation, set

$$j(t) = \mathcal{J}(\min(f^*, t)) = \int_0^t Per(\{f^* > h\}) G(|r'(h)|)\, dh.$$

Clearly,

$$\min(j(t_1), j(t_2)) \geqslant \frac{j(t_1) j(t_2)}{\mathcal{J}(f^*)}$$

and we arrive at

$$\begin{aligned}\mathcal{J}(f) - \mathcal{J}(f^*) &\geqslant \frac{\alpha_m}{R^{2m}} \int \int \mathbf{1}_{|x-y|\geqslant \beta_m R} \min\{j(f(x)), j(f(y))\}\, dx\, dy \\ &\geqslant \frac{\alpha_m}{R^{2m} \mathcal{J}(f^*)} \int \int j(f(x)) \mathbf{1}_{|x-y|\geqslant \beta_m R}\, j(f(y))\, dx\, dy.\end{aligned}$$

We conclude as in the proof of Lemma 3.1 that there exists a translation T such that

$$\mathcal{J}(f) - \mathcal{J}(f^*) \geqslant \frac{\alpha_m}{R^{2m} \mathcal{J}(f^*)} \left(\int_{|x|\geqslant \beta_m R} j(Tf(x))\, dx \right)^2.$$

The claim follows by estimating, for any $\varepsilon > 0$,

$$\int_{|x|\geqslant \beta_m R} j(Tf(x))\, dx \geqslant j(\varepsilon) \int_{|x|\geqslant \beta_m R} \mathbf{1}_{Tf(x)\geqslant \varepsilon}\, dx. \qquad \square$$

4.2. Identification of the limit

Lemma 4.3. *Let $\{f_n\}_{n\geqslant 1}$ be a sequence of nonnegative functions in $W^{1,1}(\mathbb{R}^m)$ and let $\mathcal{J}$ be a gradient functional of the form given in Eq.* (1.3), *with F strictly convex and* *increasing, and* $F(0) = 0$. *Assume that* $f_n \to f$ in L^1. *Assume furthermore that the rearrangements* f_n^* *are supported on a common ball and converge weakly to some symmetrically decreasing function* $g \in W^{1,1}$ *with* $\mathscr{J}(g) < \infty$. *If* $\mathscr{J}(f_n) \to \mathscr{J}(g)$ *then* $f \in W^{1,1}$, *and* $\mathscr{J}(f) = \mathscr{J}(g)$. *If the distribution function of g is absolutely continuous, then* $Tf = g$ *for some translation T.*

Proof. It is well known that any convex increasing function F with $F(0) = 0$ can be written in the form

$$F(t) = \int_0^t \int \mathbf{1}_{0 \leqslant \tau < h} \, dv(\tau) \, dh = \int_0^\infty \, [t - \tau]_+ \, dv(\tau),$$

where the measure v is defined on $\mathbb{R}^+$ by the derivative of F from the left,

$$v([0, h)) = F'(h).$$

Since F is strictly convex, v assigns positive weight to every interval of positive length. By assumption, $\lim \mathscr{J}(f_n) = \mathscr{J}(g)$, that is, after exchanging the order of integration

$$\lim_{n \to \infty} \int_0^\infty \int [|\nabla f_n| - \tau]_+ \, dx \, dv(\tau) = \int_0^\infty \int [|\nabla g| - \tau]_+ \, dx \, dv(\tau).$$

Since for every $\tau \geqslant 0$,

$$\varliminf_{n \to \infty} \int [|\nabla f_n| - \tau]_+ \, dx \geqslant \int [|\nabla g| - \tau]_+ \, dx$$

by Eq. (1.6), we conclude that

$$\lim_{n \to \infty} \int [|\nabla f_n| - \tau]_+ \, dx = \int [|\nabla g| - \tau]_+ \, dx \tag{4.5}$$

for almost every $\tau > 0$ at least along a subsequence (again denoted by f_n). By continuity and monotonicity in τ, Eq. (4.5) holds for all $\tau \geqslant 0$. For any $a > 0$, the sequence

$$\nabla f_n \mathbf{1}_{|\nabla f_n| \leqslant a}$$

is uniformly bounded in $L^1 \cap L^\infty$ and hence weakly compact in L^1. The remainder is bounded by

$$\int |\nabla f_n| \mathbf{1}_{|\nabla f_n| \geqslant a} \, dx \leqslant 2 \int [|\nabla f_n| - a/2]_+ \, dx \to 2 \int [|\nabla g| - a/2]_+ \quad (n \to \infty),$$

where we have used that $t \leqslant 2(t - a/2)$ for $t \geqslant a$ in the first step, and Eq. (4.5) in the second step. The last term can be made small by choosing a sufficiently large, and we conclude that the sequence ∇f_n is weakly compact in L^1. Choosing a subsequence (again denoted by f_n), we may assume that $\nabla f_n \rightharpoonup z$ weakly in L^1. By the uniqueness of weak limits, we have $\nabla f_n \rightharpoonup \nabla f$, proving that $f \in W^{1,1}$. By the continuity of the symmetric decreasing rearrangement in L^1,

$$f^* = \lim_{n\to\infty} f_n^* = g.$$

Since

$$\mathscr{I}(f^*) \leqslant \mathscr{I}(f) \leqslant \lim_{n\to\infty} \mathscr{I}(f_n) = \mathscr{I}(g),$$

it follows that $\mathscr{I}(f) = \mathscr{I}(f^*)$. If the distribution function of g is absolutely continuous, then the Brothers–Ziemer theorem implies that $Tf = g$ for some translation T [7]. $\square$

Lemma 4.4. *Let F be a strictly convex, increasing function with $F(0) = 0$. Consider a (vector-valued) sequence of functions $z_n \in L^1_{\mathrm{loc}}(R^m)$ such that z_n converges to some limit z weakly in $L^1_{\mathrm{loc}}(R^m)$. If*

$$\lim_{n\to\infty} \int_{\mathbb{R}^m} F(|z_n|)\,dx = \int_{\mathbb{R}^m} F(|z|)\,dx < \infty,$$

then

$$\lim_{n\to\infty} \int_{\mathbb{R}^m} F\left(\frac{1}{2}|z_n - z|\right) dx = 0.$$

Proof. It suffices to show that under the assumptions of the lemma, there exists a subsequence converging pointwise a.e. to z. This implies the claim by an application of Fatou's lemma to the sequence of nonnegative functions

$$\frac{F(|z_n|) + F(|z|)}{2} - F\left(\frac{|z_n - z|}{2}\right) \geqslant 0.$$

By an approximation with bounded sets, we may assume that $z_n \rightharpoonup z$ weakly in $L^1(\mathbb{R}^m)$. To show pointwise convergence, fix $a > 0$, and consider the restriction of the functions z_n to the set $\{x \in \mathbb{R}^m\colon\ |z(x)| \leqslant a\}$. It follows from the convexity of F that

$$\varliminf_{n\to\infty} \int_{|z(x)|\leqslant a} F(|z_n|)\,dx \geqslant \int_{|z(x)|\leqslant a} F(|z|)\,dx,$$

$$\varliminf_{n\to\infty} \int_{|z(x)|> a} F(|z_n|)\,dx \geqslant \int_{|z(x)|> a} F(|z|)\,dx.$$

Adding the two inequalities, we deduce from the assumption of the lemma that

$$\lim_{n\to\infty} \int_{|z(x)|\leqslant a} F(|z_n|)\,dx = \int_{|z(x)|\leqslant a} F(|z|)\,dx.$$

On the other hand, since z_n converges to z weakly in L^1,

$$\lim_{n\to\infty} \int_{|z(x)|\leqslant a} F'(|z|)\frac{z}{|z|}\cdot(z_n - z)\,dx = 0$$

everywhere. We conclude that

$$\lim_{n\to\infty} \int_{|z(x)|\leqslant a} F(|z_n|) - F(|z|) - F'(|z|)\frac{z}{|z|}\cdot(z_n - z)\,dx = 0.$$

Since the integrand is nonnegative by the convexity of F, it converges to zero pointwise almost everywhere in the region where $|z(x)|\leqslant a$. By strict convexity, the same is true for the sequence $|z_n - z|$. The proof is completed by taking $a\to\infty$. $\square$

4.3. Proof of Theorem 2

Assume for the moment that the functions f_n are uniformly bounded and that their symmetric decreasing rearrangements f_n^* are supported on the ball of radius R for some $R>0$. By Lemma 4.2, there exists a sequence of translations T_n such that for any choice of $\varepsilon>0$,

$$\mathscr{I}(f_n) - \mathscr{I}(f_n^*) \geqslant \frac{\alpha_d}{R^{2d}\mathscr{I}(f_n^*)}\left(\mathscr{I}(\min(f_n^*,\varepsilon)) \int_{|x|\geqslant\beta_d R} \mathbf{1}_{T_n f_n(x)\geqslant\varepsilon}\,dx\right)^2,$$

where α_m and β_m depend only on the dimension.

The sequence $T_n f_n$ is clearly bounded uniformly in $W^{1,1}$. It follows from the Sobolev embedding theorem that the sequence $(T_n f_n)\mathbf{1}_{|x|\leqslant\beta_m R}$ is compact in L^q for $1<q<m/(m-1)$. Moreover, since $T_n f_n$ is uniformly bounded pointwise, a simple interpolation implies that $(T_n f_n)\mathbf{1}_{|x|\leqslant\beta_m R}$ is compact in L^q for all $1<q<\infty$. Choosing a further subsequence, we may assume that $Tf_n\mathbf{1}_{|x|\leqslant\beta_m R}\to f$ in L^1. To estimate the part of $T_n f_n$ outside the ball of radius $\beta_m R$, we use that for any $\varepsilon>0$, $\mathscr{I}(\min(f_n^*,\varepsilon))\to\mathscr{I}(\min(g,\varepsilon))\neq 0$, and

$$\lim_{n\to\infty} \int_{|x|\geqslant\beta_m R} \mathbf{1}_{T_n f_n(x)>\varepsilon}\,dx = 0.$$

On the other hand,

$$\int_{|x|\geqslant\beta_m R} \mathbf{1}_{T_n f_n(x)\leqslant\varepsilon} \leqslant \int \mathbf{1}_{f_n^*(x)\leqslant\varepsilon} \leqslant \varepsilon\omega_m R^m.$$

Taking first $n \to \infty$ and then $\varepsilon \to 0$ shows that

$$\lim_{n\to\infty} ||(T_n f_n)\, \mathbf{1}_{|x|>\beta_m R}||_1 = 0,$$

thus $T_n f_n$ is compact in L^1 (and by uniform boundedness, also in L^q for $1<q<\infty$). This implies the claim in the case when $F(t) = |t|$. If F is strictly convex, then we may apply Lemma 4.3 to the sequence $T_n f_n$ to see that implies that there exists a translation T_0 such that $T_0 f = g$. We conclude with Lemma 4.4 that

$$\inf_T \mathscr{J}\left(\frac{1}{2}(Tf - g)\right) \leqslant \mathscr{J}\left(\frac{1}{2}(T_n T_0 f - g)\right) \to 0 \quad (n \to \infty).$$

This completes the proof in the case where the functions f_n are uniformly bounded and their rearrangements are supported in a common ball.

Consider now the general case of a sequence of functions f_n that satisfy the assumptions in Eqs. (1.9) and (1.10). For $R>1$ to be determined below, decompose the functions into layers, $f_n = f_n^b + f_n^u$, $g = g^b + g^u$, as in Eqs. (2.1) and (2.2). By Lemma 2.2, the functions f_n^b also satisfy the assumptions of the theorem, with g^b in place of g. By Lemma 2.1, they are uniformly bounded, and by construction, their symmetric decreasing rearrangements f_n^{*b} are supported in a common ball.

If F is strictly convex, we estimate, for any translation T,

$$\mathscr{J}\left(\frac{1}{2}(Tf_n - g)\right) \leqslant \mathscr{J}\left(\frac{1}{2}(Tf_n^b - g^b)\right) + \mathscr{J}\left(\frac{1}{2}(Tf_n^u - g^u)\right).$$

We showed in the first part of the proof that

$$\lim_{n\to\infty} \inf_T \frac{1}{2}(Tf_n^b - g^b) = 0.$$

For the second term we use

$$\overline{\lim_{n\to\infty}} \sup_T \mathscr{J}\left(\frac{1}{2}(Tf_n^u - g^u)\right) \leqslant \frac{1}{2}\left\{\overline{\lim_{n\to\infty}} \mathscr{J}(f_n^u) + \mathscr{J}(g^u)\right\} \leqslant J(g^u).$$

It follows that

$$\overline{\lim_{n\to\infty}} \inf_T \mathscr{J}\left(\frac{1}{2}(Tf_n - g)\right) \leqslant \mathscr{J}(g^u),$$

which can be made as small as we please by taking $R \to \infty$.

If $F(t) = |t|$, we have shown in the first part of the proof that there exists a sequence of translations such that f_n^b is compact in $L^{1+1/n}$ and $\nabla f_n T_n$ is tight in L^1. Moreover, as R goes to infinity, $||\nabla g^u||_1$ becomes arbitrarily small. Hence $||\nabla f_n^u||_1$ are uniformly small, which implies by Sobolev's inequality that $||f_n^u||_{1+1/n}$ are uniformly small. We thus conclude that $T_n f_n$ is compact in $L^{1+1/n}$, and $\nabla T_n f_n$ is tight in L^1. This completes the proof.

5. Applications

In this section, we illustrate how to use Theorems 1 and 2 to establish that all minimizing sequences for some variational problem converge up to the symmetries of the functional.

5.1. Dynamical stability of a gaseous star

As a first example, we will give a proof of the recent nonlinear stability results of Rein [32] on gaseous stars. Consider a self-gravitating star, as described by the compressible Euler–Poisson system:

$$\begin{aligned}
&\partial_t \rho + \nabla \cdot (\rho u) = 0, \\
&\rho \partial_t u + \rho (u \cdot \nabla) u = -\nabla P(\rho) - \rho \nabla V, \\
&\Delta V = 4\pi \rho, \qquad (5.1)
\end{aligned}$$

with the boundary condition $\lim_{|x|\to\infty} V(t,x) = 0$. Here, $\rho(t,x) \geqslant 0$ and $u(t,x) \in \mathbb{R}^3$ are the mass density and velocity field of a gaseous star at time t and position $x \in \mathbb{R}^3$, and

$$V_\rho(t,x) = -\int |x-y|^{-1} \rho(t,y)\, dy, \qquad (5.2)$$

is the corresponding gravitational potential. For simplicity, we assume that the pressure is given by $P(\rho) = \rho^\gamma$. The energy functional

$$\mathscr{E}(\rho, u) = \frac{1}{2} \int |u|^2 \rho \, dx + \frac{1}{\gamma - 1} \int \rho^\gamma \, dx - \frac{1}{2} \int \int \rho(x) |x-y|^{-1} \rho(y)\, dx\, dy,$$

is formally conserved under the motion generated by Eq. (5.1). The first term in the energy functional represents the kinetic energy, the second term is the contribution of the pressure, and the third term is the gravitational potential energy. A family of steady states is obtained by minimizing the time-independent functional

$$\mathscr{H}(\rho) = \frac{1}{\gamma - 1} \int \rho^\gamma \, dx - \frac{1}{2} \int \int \rho(x) |x-y|^{-1} \rho(y)\, dx\, dy \qquad (5.3)$$

subject to the mass constraint $\int \rho(x)\, dx = M$. A symmetric minimizer is given by

$$\rho_0(x) = c(\gamma)[E_0 - V_{\rho_0}(x)]_+^{1/(\gamma-1)}, \qquad (5.4)$$

where $E_0 \leqslant 0$ is a Lagrange multiplier associated with the mass constraint, and $V_{\rho_0}(x)$ is the potential induced by ρ_0 through Eq. (5.2). The minimizer is unique up to translation. The main result in [32] is the following.

Theorem (Rein [32]). *For $\gamma > 4/3$, the symmetric steady-state solution $\rho_0(x)$ is dynamically stable up to translations, among possible weak solutions which satisfy the mass constraint and whose energy does not exceed the energy of the initial values.*

Here, the distance from ρ_0 is measured by

$$d(\rho, \rho_0) = \frac{1}{\gamma - 1} \int \rho^\gamma - \rho_0^\gamma + (V_{\rho_0} - E_0)(\rho - \rho_0)\, dx.$$

Notice that since $\gamma > 4/3$, the integrand above is nonnegative, by a Taylor expansion around $\rho_0(x)$ in (5.4). The crucial part is to establish that for any minimizing sequence ρ_n, there exists a sequence of translations T_n on $\mathbb{R}^3$ such that

$$||\nabla V_{T_n \rho_n} - \nabla V_{\rho_0}||_2 \to 0, \tag{5.5}$$

see Theorem 1 in [32], and similar arguments for stable galaxy configurations in [14–18,30].

Proof of Theorem. Denote by

$$\mathscr{I}(\rho) = \int \int \rho(x)|x-y|^{-1}\rho(y)\, dx\, dy = ||\nabla V_\rho||_2^2,$$

the gravitational potential energy associated with the mass distribution ρ.

Step 1: The compactness of symmetric minimizing sequences follows from [32, Lemma 4.1]. It is shown there that (5.5) holds with no translations needed for any symmetric minimizing sequence, that is,

$$\lim_{n \to \infty} \mathscr{I}(\rho_n - \rho_0) = 0.$$

As a matter of fact, the splitting and scaling argument used in the proof leads to an a priori estimate for the radius of $\rho_0(x)$, of the form $|x| \leqslant \frac{3M^2}{5h_M}$, with an explicit constant h_M.

Step 2: Given a general minimizing sequence ρ_n with $\lim_{n\to\infty} \int \rho_n = M$. Using the equimeasurability of ρ_n with ρ_n^* and the Riesz rearrangement inequality, we see that the sequence of symmetrizations ρ_n^* is again a minimizing sequence, and that

$$\lim_{n \to \infty} \mathscr{I}(\rho_n) = \lim_{n \to \infty} \mathscr{I}(\rho_n^*) = \mathscr{I}(\rho_0).$$

By Step 1,

$$\lim_{n \to \infty} \mathscr{I}(\rho_n^* - \rho_0) = 0.$$

Since the Coulomb kernel $K(x-y) = |x-y|^{-1}$ is strictly symmetrically decreasing and positive definite, Eq. (5.5) follows directly from Theorem 1. $\square$

5.2. Stability in galactic dynamics

As a further illustration, we present an argument for the stability of symmetric steady states in galactic dynamics which was communicated to us by Rein [31]. Consider a large ensemble of stars (e.g. a galaxy) interacting by the gravitational field that they create collectively. In contrast to the gaseous star problem in the last section, it is impossible now to study the dynamics of each individual star. The most fundamental physical model for describing the dynamics of a galaxy is based on kinetic theory, in which the ensemble is described by a phase space density $f(t,x,v)$ rather than by the particle density and velocity field. Here $(x,v)\in\mathbb{R}^3\times\mathbb{R}^3$ denote the position and (independent) momentum variables. In astrophysics the dynamics of typical galaxies or globular clusters is then described by the Vlasov–Poisson system.

$$\begin{aligned}\partial_t f + v\cdot\nabla_x f - \nabla_x V\cdot\nabla_v f &= 0,\\ \Delta V &= 4\pi\rho,\end{aligned} \tag{5.6}$$

where

$$\rho(t,x) = \int f(t,x,v)\,dv \tag{5.7}$$

is the particle density corresponding to f, and the gravitational potential V again satisfies Eq. (5.2). The sum of the kinetic and potential energies

$$\begin{aligned}\mathscr{E}(f) &= \mathscr{E}_{\mathrm{kin}}(f) + \mathscr{E}_{\mathrm{pot}}(\rho)\\ &= \frac{1}{2}\int\int |v|^2 f(x,v)\,dv\,dx - \frac{1}{2}\int\int \rho(x)|x-y|^{-1}\rho(y)\,dx\,dy\end{aligned}$$

is conserved under the dynamical system generated by Eqs. (5.6). The rare collisions between stars are neglected in such a model. As a consequence, the Vlasov–Poisson system has an additional scaling symmetry and a continuum of conserved quantities given by the so-called Casimir functionals

$$\mathscr{C}(f) := \int\int \Phi(f(x,v))\,dv\,dx,$$

where Φ is a convex function satisfying appropriate growth conditions. For simplicity, we assume here that $\Phi(f) = f^{1+1/k}$ with $0<k<3/2$. Steady states can be obtained by minimizing

$$\mathscr{C}(f) + \mathscr{E}(f) \tag{5.8}$$

under the constraint that the total mass $\int\int f\,dv\,dx = M$ is a prescribed positive constant.

The machinery of the present paper does not apply directly to the variational problem in Eq. (5.8). It was pointed out by Rein [31] that the problem can be reduced to one in terms of the particle density $\rho = \rho(x)$. We gratefully reproduce his argument here; details can be found in [30].

The idea is to perform the minimization problem in Eq. (5.8) in two stages,

$$\inf_{f:\ \int\int f(x,v)\,dv\,dx = M} \{\mathscr{C}(f) + \mathscr{E}(f)\}$$
$$= \inf_{\rho:\ \int \rho(x)\,dx = M} \left\{ \inf_{f:\ \int f(\cdot,v)\,dv = \rho} \{\mathscr{C}(f) + \mathscr{E}_{\rm kin}(f)\} + \mathscr{E}_{\rm pot}(\rho) \right\}.$$

The inner minimization amounts to computing for a given particle density ρ the composition $\Psi \circ \rho$, where for $r \geqslant 0$

$$\Psi(r) = \inf\left\{ \int \Phi(g(v)) + \frac{1}{2}|v|^2 g(v)\,dv \mid 0 \leqslant g \in L^1(\mathbb{R}^3),\ \int g(v)\,dv = r \right\}. \tag{5.9}$$

By the strict convexity of Φ, the minimizer in Eq. (5.9) is uniquely determined by r, and thus any minimizing phase space density for Eq. (5.8) is uniquely determined by the corresponding particle density. The relationship between Φ and Ψ can be made explicit by noting that their Legendre transforms $\hat{\Phi}$ and $\hat{\Psi}$ satisfy $\hat{\Psi}(t) = \int \hat{\Phi}(t - |v|^2/2)\,dv$. In particular, for $\Phi(f) = f^{1+1/k}$ we find that up to a multiplicative constant $\Psi(\rho) = \rho^\gamma$ with $\gamma = 1 + 1/(k + 3/2) \in (4/3, 5/3)$.

The outer minimization problem is thus reduced to minimizing

$$\mathscr{H}(\rho) = \int \Psi(\rho(x))\,dx - \int\int \rho(x)|x-y|^{-1}\rho(y)\,dx\,dy \tag{5.10}$$

over particle densities ρ satisfying the mass constraint $\int \rho(x)\,dx = M$. This problem has precisely the form of Eq. (5.3) considered in Section 5.1. In particular, there exist symmetric steady states with the particle density given by Eq. (5.4). The corresponding symmetric minimizing phase space density is given by

$$f_0(x,v) = [E_0 - |v|^2/2 - V_{\rho_0}(x)]_+^k, \quad 0 < k < 3/2.$$

We claim that from the point of view of stability for the Vlasov–Poisson system all the relevant knowledge for the variational problem in Eq. (5.8) can be extracted from its reduced form in Eq. (5.10). To see this, let f_n be a minimizing sequence for Eq. (5.8), and let ρ_n be the corresponding sequence of particle densities determined by Eq. (5.7). Since ρ_n is a minimizing sequence for the reduced problem in Eq. (5.10), we conclude from Section 5.1 that ρ_n converges (up to suitable translations T_n) to some particle density ρ_0 and $\nabla V_{T_n\rho_n} \to \nabla V_{\rho_0}$ in L^2. Choosing a subsequence and using the special form of Φ, we may assume that the sequence of phase space densities $T_n f_n$ converges weakly in $L^{1+1/k}$ to some limiting function f_0. Since $\nabla V_{T_n f_n}$ is compact in L^2, the energy-Casimir functional $\mathscr{E} + \mathscr{C}$ is lower semicontinuous, and its values must converge along the sequence, and we conclude that $T_n f_n \to f_0$ strongly in $L^{1+1/k}$. It follows that f_0 is the unique minimizer for the full problem in Eq. (5.8) determined by $\rho_0(x)$. In summary, there exists a sequence of translations T_n such that $T_n f_n \to f_0$.

5.3. Maximizing sequences for the HLS functional

We will show how to use Theorem 1 to verify that all maximizing sequences for the Hardy–Littlewood–Sobolev inequality converge up to scalings, translations, and phase factors, as first proved by Lions [25,26]. The Hardy–Littlewood–Sobolev inequality states that

$$\mathscr{I}(f) := \int\int f(x)|x-y|^{-\lambda}\bar{f}(y)\,dx\,dy \leqslant I(m,p)||f||_p^2, \quad \frac{2}{p}+\frac{\lambda}{m}=2$$

for any complex-valued f in $L^p(\mathbb{R}^m)$. Both the functional $\mathscr{I}$ and the p-norm are invariant under the translation by vectors $a \in \mathbb{R}^m$ and scaling by factors $\sigma > 0$:

$$Tf(x) = f(x-a), \quad Sf(x) = \sigma^{-m/p} f(x/\sigma).$$

The sharp constant

$$I(m,p) = \sup_{||f||_p^p=1} \mathscr{I}(f) \tag{5.11}$$

was determined by Lieb in [22]. It is achieved for

$$g(x) = \left(\frac{2}{1+|x|^2}\right)^{m/p}; \tag{5.12}$$

in fact, g is the unique symmetrically decreasing optimizer of Eq. (5.11) with $\int_{|x|<1} g(x)^p\,dx = 1/2$ [24, Theorem 4.3 and Lemma 4.8].

Lieb's identification of the optimizers combines the conformal invariance of Eq. (5.11) and the sharp Riesz rearrangement inequality with a subtle compactness argument. The most direct proof of the sharp Hardy–Littlewood–Sobolev inequality uses the competing symmetries technique to construct special maximizing sequences with good convergence properties, thus sidestepping the compactness issue [2,11], (see [24, Theorem 4.6]). In fact, all maximizing sequences for Eq. (5.11) converge to g up to suitable scalings, translations, and multiplication by phase factors:

Theorem (Lions [26]). *For every sequence of functions f_n on $\mathbb{R}^m$ satisfying*

$$||f_n||_p = 1, \quad \lim_{n\to\infty} \mathscr{I}(f_n) = \mathscr{I}(g),$$

where g is given by Eq. (5.12), *there exist sequences of scalings S_n, translations T_n, and phase factors $e^{i\phi_n}$ such that*

$$\lim_{n\to\infty} \mathscr{I}(e^{i\phi_n}T_nS_nf_n - g) = 0, \quad \lim_{n\to\infty} ||e^{i\phi_n}T_nS_nf_n - g) = 0.$$

Proof. *Step* 1: Although it is not explicitly stated there, Lieb shows in his proof of the maximality of g that every maximizing sequence of symmetrically decreasing functions g_n for converges to g up to scalings [22, p. 536]. In other words, there exists a sequence of scalings S_n such that

$$\lim_{n\to\infty} ||S_ng_n - g||_p = 0.$$

Since $\mathscr{I}$ is continuous in L^p by the (non-sharp) Hardy–Littlewood–Sobolev inequality, it follows that

$$\lim_{n\to\infty} \mathscr{I}(S_ng_n - g) = 0.$$

The compactness of symmetric minimizing sequences up to scaling can also be shown directly, by using the splitting and scaling technique developed in [15].

Step 2: Consider a general maximizing sequence of nonnegative functions f_n. Clearly f_n^* is again a maximizing sequence. By Step 1, there exists a sequence of scalings such that

$$\lim_{n\to\infty} \mathscr{I}(S_nf_n^* - g) = 0.$$

Since f_n is a maximizing sequence, we have

$$\lim_{n\to\infty} \mathscr{I}(f_n) = \lim_{n\to\infty} \mathscr{I}(f_n^*) = \mathscr{I}(g).$$

The kernel $K(x-y) = |x-y|^{-\lambda}$ is positive definite and symmetrically decreasing, and we may apply Theorem 1 to the sequence S_nf_n to obtain a sequence of translations such that

$$\lim_{n\to\infty} \mathscr{I}(T_nS_nf_n - g) = 0,$$

in particular, $T_nS_nf_n \to g$ pointwise almost everywhere at least along suitable subsequences. Since $\lim_{n\to\infty} ||f_n||_p = ||g||_p$, it follows from the characterization of the missing term in Fatou's lemma that

$$\lim_{n\to\infty} ||T_nS_nf_n - g||_p = 0.$$

Conclusion: For a general maximizing sequence of real-valued functions, it is easy to see that there exists a subsequence along which either the positive parts $[f_n]_+$ or the negative parts $[f_n]_-$ form again a maximizing sequence, and that the other part converges to zero. Similarly, Schwarz' inequality implies that the real and imaginary parts of a complex-valued sequence are again optimizing sequences, and that their ratio converges to a constant. □

5.4. Minimizing sequences for the Sobolev constant

Finally, we show the corresponding compactness result for minimizing sequences of the Sobolev inequality . The Sobolev inequality bounds the norm of a function in $L^p(\mathbb{R}^m)$ by a corresponding gradient norm,

$$\mathscr{J}(f) := \int |\nabla f|^p \, dx \geqslant J(m,p)||f||_{p^*}^p, \quad p^* = \frac{mp}{m-p}, \quad 1 \leqslant p < m.$$

The functional and the p^*-norm are invariant under translation by vectors $a \in \mathbb{R}^m$ and scaling by dilation factors $\sigma > 0$

$$Tf(x) = f(x-a), \quad Sf(x) = \sigma^{-m/p^*} f(x/\sigma).$$

The sharp constant

$$J(m,p) = \inf_{||f||_{p^*}^{p^*}=1} \mathscr{J}(f) \tag{5.13}$$

was determined by Talenti [37] and Aubin [1]. For $p > 1$ it is assumed for the function

$$g_{\alpha,\beta}(x) = \left(\frac{1}{\alpha + \beta ||x||^{p/p-1}} \right)^{p^*/p}, \tag{5.14}$$

where α and β are positive constants determined by the values of $||g||_{p^*}$ and $\int_{|x|<1} g^{p^*}$. For $p = 1$, $J(m,1)$ is the isoperimetric constant in $\mathbb{R}^m$, which is assumed not in $W^{1,1}$ but by the characteristic function of a ball in BV. The optimizer is unique up to scaling, translation, and multiplication by constants.

In the proof, Talenti uses the rearrangement inequality for convex gradient functionals and Aubin uses the isoperimetric inequality to reduce the variational problem to radially decreasing functions. Then they analyze the ordinary differential equation associated with the resulting one-dimensional problem. In the special case $p = 2$, Eq. (5.13) is again conformally invariant, and the competing symmetries technique quickly yields the optimizers. A recent proof, using optimal transportation techniques, avoids compactness issues altogether [13]. We will give a proof that for $p > 1$, all minimizing sequences converge up to scalings, translation, and multiplication by phase factors. In the case $p = 1$, the minimizer is a function of bounded variation, but the minimizing sequence still has some tightness properties.

Theorem (Lions [26]). *Given a sequence of functions $f_n \in W^{1,p}(\mathbb{R}^m)$ with*

$$||f_n||_p = 1, \quad \lim_{n\to\infty} \mathscr{J}(f_n) = J(m,p).$$

1. *If $p>1$, there exist sequences of scalings, S_n, translations T_n, and phase factors $e^{i\phi_n}$ such that the sequence defined by satisfies*

$$\lim_{n\to\infty} ||e^{i\phi_n}T_nS_nf_n - g||_p = 0, \quad \lim_{n\to\infty} \mathscr{J}(e^{i\phi_n}T_nS_nf_n - g) = 0.$$

2. *If $p=1$, there exist sequences of scalings S_n, translations T_n, and phase factors $e^{i\phi_n}$ such that the sequence of gradients $\nabla\{e^{i\phi_n}T_nS_nf_n\}$ is tight in L^1 and the sequence $e^{i\phi_n}T_nS_nf_n$ is compact in $L^{\frac{n}{n-1}}$.*

Proof. *Step* 1: Let g_n be a sequence of symmetrically decreasing functions with $||g_n||_{p^*} = 1$ and $\lim \mathscr{J}(g_n) = J(m,p)$. By scaling, we may assume that

$$\int_{|x|\leqslant 1} g_n^{p^*} = \int_{|x|\geqslant 1} g_n^{p^*} = \frac{1}{2}.$$

Choosing a subsequence, we may assume that g_n converges weakly in $W^{1,p}$ (or in BV if $p=1$), and in L^{p^*} to some symmetrically decreasing limit function $g \in W^{1,p}$. Since the g_n are symmetrically decreasing, they also converge pointwise almost everywhere. Clearly, $||g||_{p^*}^{p^*} \leqslant 1$ and $\mathscr{J}(g) \leqslant J(m,p)$.

We want to show that the sequence g_n can concentrate neither at $|x|=0$ nor at $|x|=\infty$. Let $\mathscr{X}$ be a symmetrically decreasing smooth cutoff function with values in $[0,1]$, satisfying $\mathscr{X}(x)=1$ for $|x|<1$ and $\mathscr{X}(x)=0$ for $|x|>2$. For $R>2$, we split g_n into three parts,

$$g^\ell(x) = \mathscr{X}(Rx)g(x), \quad g^r(x) = \mathscr{X}(x/R)g(x), \quad g^c = g - g^\ell - g^r$$

and correspondingly for the functions g_n. It follows from the uniform bounds in Lemma 2.1 and the pointwise convergence that $g_n^c \to g^c$ strongly in L^q for all $q \geqslant 1$, and that $g_n^\ell \to g$ strongly in L^q for all $q<p^*$. Let

$$\theta_n^\ell(R) = ||g_n^\ell||_{p^*}^{p^*}, \quad \theta_n^r(R) = ||g_n^r||_{p^*}^{p^*}.$$

We compute

$$\mathscr{J}(g_n) = \int |\nabla g_n|^p \, dx \geqslant \int |(\mathscr{X}(x/R) - \mathscr{X}(Rx))\nabla g_n|^p \, dx + \int |\mathscr{X}(Rx)\nabla g_n|^p \, dx + \int |(1-\mathscr{X}(x/R))\nabla g_n|^p \, dx. \tag{5.15}$$

Using the product rule and the definition of $\mathscr{J}$, the first term on the right-hand side is estimated by

$$\begin{aligned}\int |(\mathscr{X}(x/R) - \mathscr{X}(Rx))\nabla g_n|^p\, dx &\geqslant \mathscr{J}(g_n^c) - \int (R|\nabla\mathscr{X}(Rx)| + R^{-1}|\nabla\mathscr{X}(x/R)|)g_n\, dx \\ &\geqslant (1-\theta_n^\ell(R) - \theta_n^r(T))^{p/p^*} J(m,p) \\ &\quad - C(R^{1/p^*} + 2^m\omega_m g_n(R)),\end{aligned}$$

where the constant C depends only on the cutoff function $\mathscr{X}$. We have used the definition of the sharp Sobolev constant $J(m,p)$ to estimate the first term, Hölder's inequality for the second, and the fact that g_n is symmetrically decreasing for the third. The second and third terms on the right-hand side of Eq. (5.15) are similarly bounded below by

$$\int |\mathscr{X}(Rx)\nabla g_n|^p\, dx \geqslant (\theta_n^\ell(R))^{p/p^*} J(m,p) - CR^{1/p^*},$$

$$\int |(1-\mathscr{X}(x/R))\nabla g_n|^p\, dx \geqslant (\theta_n^r(R))^{p/p^*} J(m,p) - C2^m\omega_m g(2R).$$

Inserting these estimates into Eq. (5.15) and taking limits, we deduce that

$$\lim_{n\to\infty}\{1 - [(1-\theta_n^\ell(R) - \theta_n^r(R))^{p/p^*} + (\theta_n^\ell(R))^{p/p^*} + (\theta_n^r(R))^{p/p^*}]\} \to 0 \quad (R\to\infty).$$

We have used that $\lim \mathscr{J}(g_n) = J(m,p)$, and that g_n converges to g pointwise. Since $\theta_n^\ell(R) \leqslant 1/2$ and $\theta_n^r(R) \leqslant 1/2$ for all $R>2$ by our choice of scaling, the strict convexity of the function $t\to t^{p/p^*}$ implies that

$$\overline{\lim_{n\to\infty}}\{\theta_n^\ell(R) + \theta_n^r(R)\} \to 0 \quad (R\to 0).$$

It follows that $g_n \to g$ strongly in L^{p^*}, and consequently $||g||_{p^*} = 1$. By the definition of the optimal constant $J(m,p)$ and Fatou's lemma, we have $||\nabla g||_p^p = \mathscr{J}(g) = J(m,p)$, and ∇g_n converges to ∇g strongly in L^p. Thus g is an extremal for the Sobolev inequality, and is given by Eq. (5.14), with the scaling parameters α and β determined by

$$||g||_{p^*} = 1, \quad \int_{|x|<1} g^{p^*} = \frac{1}{2}.$$

Since all suitably scaled subsequences converge to the same limit, the entire sequence converges to g in $L^p (p>1)$, as claimed. For $p=1$ we use that $||g||_{p^*}^{p^*} = 1$ and $\lim \mathscr{J}(g_n) = J(1,p)$ which $\nabla g_n \to \nabla g$ weakly in measure.

Step 2: Consider a minimizing sequence of nonnegative functions f_n. Clearly the symmetric decreasing rearrangements f_n^* form again a minimizing sequence, If $p>1$, by Step 1, there exists a sequence of scalings S_n such that $\lim_{n\to\infty} ||S_n f_n^* - g||_{p^*} =$

0, $\lim_{n\to\infty} \mathscr{J}(S_n f_n^* - g) = 0$. For $p > 1$, the limiting function g is strictly symmetrically decreasing, strictly positive, and has a continuous distribution function. By Theorem 2 applied to $S_n f_n$, there exists a sequence of translations T_n such that

$$\lim_{n\to\infty} \mathscr{J}(T_n S_n f_n - g) = 0.$$

It follows from the Sobolev inequality that

$$\lim_{n\to\infty} ||T_n S_n f_n - g||_{p^*} = 0.$$

On the other hand, if $p = 1$, we then have $\mathscr{J}(S_n f_n^*) \to \mathscr{J}(g)$ and $\lim_{n\to\infty} ||S_n f_n^* - g||_{p^*} = 0$, and we can apply the second part of Theorem 2.

Conclusion. For a general complex-valued minimizing sequence, the claim follows by splitting the sequence into its real and imaginary parts and using the convexity inequality for gradients [24, Theorem 7.8]. □

Acknowledgments

The research stems from an extensive collaboration of Y.G. with Gerhard Rein on stellar dynamics. We thank Robert McCann for initiating our collaboration, Richard Laugesen for introducing us to asymmetry inequalities, and Gerhard Rein for contributing the application to galactic dynamics discussed in Section 5.2. A.B. gratefully acknowledges the hospitality of Princeton University during the Spring 2002 semester, and thanks Elliott Lieb for several useful conversations and the idea for the proof of Lemma 2.5. The work was supported in part by NSF Grants DMS-0305161 and DMS-0308040 and two Sloan research fellowships.

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